

The Isomorphism Relation Between Tree-Automatic Structures

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Abstract

An ω -tree-automatic structure is a relational structure whose domain and relations are accepted by Muller or Rabin tree automata. We investigate in this paper the isomorphism problem for ω -tree-automatic structures. We prove first that the isomorphism relation for ω -tree-automatic boolean algebras (respectively, partial orders, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $n \geq 2$) is not determined by the axiomatic system **ZFC**. Then we prove that the isomorphism problem for ω -tree-automatic boolean algebras (respectively, partial orders, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $n \geq 2$) is neither a Σ_2^1 -set nor a Π_2^1 -set.

Keywords: ω -tree-automatic structures; boolean algebras; partial orders; rings; groups; isomorphism relation; models of set theory; independence results.

1 Introduction

An automatic structure is a relational structure whose domain and relations are recognizable by finite automata reading finite words. Automatic

structures have very nice decidability and definability properties and have been much studied in the last few years, see [BG04, KNRS07, Nie07, Rub04, Rub08]. They form a subclass of the class of (countable) recursive structures where “recursive” is replaced by “recognizable by finite automata”. Blumensath considered in [Blu99] more powerful kinds of automata. If we replace automata by tree automata (respectively, Büchi automata reading infinite words, Muller or Rabin tree automata reading infinite labelled trees) then we get the notion of tree-automatic (respectively, ω -automatic, ω -tree-automatic) structures. In particular, an ω -automatic or ω -tree-automatic structure may have uncountable cardinality. All these kinds of automatic structures have the two following fundamental properties. (1) The class of automatic (respectively, tree-automatic, ω -automatic, ω -tree-automatic) structures is closed under first-order interpretations. (2) The first-order theory of an automatic (respectively, tree-automatic, ω -automatic, ω -tree-automatic) structure is decidable.

A natural problem is to classify firstly automatic structures (presentable by finite automata) using some invariants. For instance Delhommé proved that the automatic ordinals are the ordinals smaller than ω^ω , see [Del04, Rub04, Rub08]. And Khoushainov, Nies, Rubin, and Stephan proved in [KNRS07] that the automatic infinite boolean algebras are the finite products $B_{fin-cof}^n$ of the boolean algebra $B_{fin-cof}$ of finite or cofinite subsets of the set of positive integers \mathbb{N} . On the other hand some classes of automatic structures, like automatic linear orders, or automatic groups, are not completely determined. Another fundamental question which naturally arises in the investigation of the richness of the class of automatic structures is the following: “what is the complexity of the isomorphism problem for the class of automatic structures, or for a subclass of it?” Khoushainov, Nies, Rubin, and Stephan proved in [KNRS07] that the isomorphism problem for the class of automatic structures, or even for the class of automatic graphs, is Σ_1^1 -complete, i.e. as complicated as the isomorphism problem for recursive structures. However for some classes of automatic structures, like the classes of automatic ordinals or of automatic boolean algebras, the isomorphism problem is decidable, [KNRS07, Rub08]. But for other classes like the classes of automatic linear orders or groups the complexity or even the decidability of the isomorphism problem is still unknown.

There has been less classification work for ω -automatic and ω -tree-automatic structures. In particular, it seems that no complete classification exists for classes of ω -automatic or ω -tree-automatic structures, like the result classifying completely the automatic ordinals or the automatic boolean algebras.

Some foundational questions about ω -automatic structures have been recently solved. Kuske and Lohrey proved in [KL08] that the first-order theory, extended with some cardinality quantifiers, of an (injectively) ω -automatic structure is decidable. Next Barany, Kaiser and Rubin extended this result to all ω -automatic structures and proved that an ω -automatic structure which is countable is automatic, i.e. presentable by automata reading finite words, [BKR08]. One of the most important foundational problems in this area is again the question of the complexity of the isomorphism problem for ω -automatic or ω -tree-automatic structures. In a recent paper Hjorth, Khoussainov, Montalbán, and Nies proved that the isomorphism problem for ω -automatic structures is not a Σ_2^1 -set, [HKMN08]. In fact their proof implies also that this isomorphism problem is not a Π_2^1 -set. Moreover this is also the case for the restricted class of ω -automatic (abelian) groups and for the class of all ω -tree-automatic structures which is an extension of the class of ω -automatic structures.

We investigate in this paper the isomorphism problem for some classes of ω -tree-automatic structures. We prove first that the isomorphism relation for ω -tree-automatic structures (respectively, ω -tree-automatic boolean algebras) is not determined by the axiomatic system ZFC. Indeed, using known results about quotients of the boolean algebra $\mathcal{P}(\mathbb{N})$ over analytic ideals on \mathbb{N} , we prove that there exist two ω -tree-automatic boolean algebras \mathcal{B}_1 and \mathcal{B}_2 such that: (1) \mathcal{B}_1 is isomorphic to \mathcal{B}_2 in (ZFC + CH) and (2) \mathcal{B}_1 is not isomorphic to \mathcal{B}_2 in (ZFC + OCA), where the axioms CH, OCA denote respectively the Continuum Hypothesis, the Open Coloring Axiom. Then we infer from this result that the isomorphism relation for ω -tree-automatic partial orders (respectively, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $n \geq 2$) is not determined by the axiomatic system ZFC. This shows the importance of different axiomatic systems of Set Theory in the area of ω -tree-automatic structures. Then we prove that the isomorphism problem for ω -tree-automatic boolean algebras (respectively, partial orders, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $n \geq 2$) is neither a Σ_2^1 -set nor a Π_2^1 -set.

The paper is organized as follows. In Section 2 we recall definitions and first properties of ω -automatic and ω -tree-automatic structures. We give in Section 3 the two boolean algebras \mathcal{B}_1 and \mathcal{B}_2 and prove that they are ω -tree-automatic atomless boolean algebras. In Section 4 we introduce notions of topology and prove some topological properties which will be useful in the sequel. We recall some notions of Set Theory in Section 5. We prove our

main results in Section 6.

2 ω -tree-automatic structures

When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \dots a_k$, where $a_i \in \Sigma$ for $i = 1, \dots, k$, and k is an integer ≥ 1 . The *length* of x is k . The *empty word* has no letter and is denoted by ε ; its length is 0. For $x = a_1 \dots a_k$, we write $x(i) = a_i$. Σ^* is the *set of finite words* (including the empty word) over Σ .

The *first infinite ordinal* is ω . An ω -word over Σ is an ω -sequence $a_1 \dots a_n \dots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When σ is an ω -word over Σ , we write $\sigma = \sigma(1)\sigma(2) \dots \sigma(n) \dots$, where for all i , $\sigma(i) \in \Sigma$.

The *set of ω -words* over the alphabet Σ is denoted by Σ^ω . An ω -language over an alphabet Σ is a subset of Σ^ω .

We consider in this paper relational structures which are presentable by automata reading infinite trees or infinite words. We assume that the reader is familiar with the notion of Büchi automaton reading infinite words over a finite alphabet which can be found for instance in [Tho90, Sta97]. Informally speaking an ω -word x over Σ is accepted by a Büchi automaton \mathcal{A} iff there is an infinite run of \mathcal{A} on x entering infinitely often in some final state of \mathcal{A} . The ω -language $L(\mathcal{A}) \subseteq \Sigma^\omega$ accepted by the Büchi automaton \mathcal{A} is the set of ω -words x accepted by \mathcal{A} .

We introduce now languages of infinite binary trees whose nodes are labelled in a finite alphabet Σ .

A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where r means “right” and l means “left”. Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^* \rightarrow \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T_Σ^ω . A tree language is a subset of T_Σ^ω , for some alphabet Σ . (Notice that we shall only consider in the sequel *infinite* trees so we shall often simply call tree an *infinite* tree).

Let t be a tree. A branch B of t is a subset of the set of nodes of t which is linearly ordered by the tree partial order \sqsubseteq and which is closed under prefix relation, i.e. if x and y are nodes of t such that $y \in B$ and $x \sqsubseteq y$ then $x \in B$.

A branch B of a tree is said to be maximal iff there is not any other branch of t which strictly contains B .

Let t be an infinite binary tree in T_Σ^ω . If B is a maximal branch of t , then this branch is infinite. Let $(u_i)_{i \geq 0}$ be the enumeration of the nodes in B which is strictly increasing for the prefix order.

The infinite sequence of labels of the nodes of such a maximal branch B , i.e. $t(u_0)t(u_1) \dots t(u_n) \dots$ is called a path. It is an ω -word over the alphabet Σ .

We are now going to define tree automata and regular tree languages.

Definition 2.1 A (nondeterministic) tree automaton is a quadruple $\mathcal{A} = (Q, \Sigma, \Delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state and $\Delta \subseteq Q \times \Sigma \times Q \times Q$ is the transition relation. A run of the tree automaton \mathcal{A} on an infinite binary tree $t \in T_\Sigma^\omega$ is an infinite binary tree $\rho \in T_Q^\omega$ such that:

(a) $\rho(\varepsilon) = q_0$ and (b) for each $u \in \{l, r\}^*$, $(\rho(u), t(u), \rho(u.l), \rho(u.r)) \in \Delta$.

Definition 2.2 A Muller (nondeterministic) tree automaton is a 5-tuple $\mathcal{A} = (Q, \Sigma, \Delta, q_0, \mathcal{F})$, where (Q, Σ, Δ, q_0) is a tree automaton and $\mathcal{F} \subseteq 2^Q$ is the collection of designated state sets. A run ρ of the Muller tree automaton \mathcal{A} on an infinite binary tree $t \in T_\Sigma^\omega$ is said to be accepting if for each path p of ρ , the set of states appearing infinitely often on this path is in \mathcal{F} . The tree language $L(\mathcal{A})$ accepted by the Muller tree automaton \mathcal{A} is the set of infinite binary trees $t \in T_\Sigma^\omega$ such that there is (at least) one accepting run of \mathcal{A} on t . A tree language $L \subseteq T_\Sigma^\omega$ is regular iff there exists a Muller automaton \mathcal{A} such that $L = L(\mathcal{A})$.

Remark 2.3 A tree language is accepted by a Muller tree automaton iff it is accepted by some Rabin tree automaton. We refer for instance to [Tho90, PP04] for the definition of Rabin tree automaton.

We now recall some fundamental closure properties of regular ω -languages and of regular tree languages.

Theorem 2.4 (see [Tho90, PP04]) The class of regular ω -languages (respectively, of regular tree languages) is effectively closed under finite union, finite intersection, and complementation, i.e. we can effectively construct, from two Büchi automata (respectively, Muller tree automata) \mathcal{A} and \mathcal{B} , some Büchi automata (respectively, Muller tree automata) \mathcal{C}_1 , \mathcal{C}_2 , and \mathcal{C}_3 , such that $L(\mathcal{C}_1) = L(\mathcal{A}) \cup L(\mathcal{B})$, $L(\mathcal{C}_2) = L(\mathcal{A}) \cap L(\mathcal{B})$, and $L(\mathcal{C}_3)$ is the complement of $L(\mathcal{A})$.

Notice that one can consider a relation $R \subseteq \Sigma_1^\omega \times \Sigma_2^\omega \times \dots \times \Sigma_n^\omega$, where $\Sigma_1, \Sigma_2, \dots, \Sigma_n$, are finite alphabets, as an ω -language over the product alphabet $\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$. In a similar way we can consider a relation $R \subseteq T_{\Sigma_1}^\omega \times T_{\Sigma_2}^\omega \times \dots \times T_{\Sigma_n}^\omega$, as a tree language over the product alphabet $\Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$.

Let now $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq n})$ be a relational structure, where M is the domain, and for each $i \in [1, n]$ R_i^M is a relation of finite arity n_i on the domain M . The structure is said to be ω -automatic (respectively, ω -tree-automatic) if there is a presentation of the structure where the domain and the relations on the domain are accepted by Büchi automata (respectively, by Muller tree automata), in the following sense.

Definition 2.5 (see [Blu99]) *Let $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq n})$ be a relational structure, where $n \geq 1$ is an integer, and each relation R_i is of finite arity n_i . An ω -tree-automatic presentation of the structure \mathcal{M} is formed by a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq n})$, and a mapping h from $L(\mathcal{A})$ onto M , such that:*

1. *The automaton $\mathcal{A}_=$ accepts an equivalence relation E_\equiv on $L(\mathcal{A})$, and*
2. *For each $i \in [1, n]$, the automaton \mathcal{A}_i accepts an n_i -ary relation R'_i on $L(\mathcal{A})$ such that E_\equiv is compatible with R'_i , and*
3. *The mapping h is an isomorphism from the quotient structure $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq n})/E_\equiv$ onto \mathcal{M} .*

The ω -tree-automatic presentation is said to be injective if the equivalence relation E_\equiv is just the equality relation on $L(\mathcal{A})$. In this case $\mathcal{A}_=$ and E_\equiv can be omitted and h is simply an isomorphism from $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq n})$ onto \mathcal{M} . A relational structure is said to be (injectively) ω -tree-automatic if it has an (injective) ω -tree-automatic presentation.

Notice that sometimes an ω -tree-automatic presentation is only given by a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq n})$, i.e. *without the mapping h* . In that case we still get the ω -tree-automatic structure $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq n})/E_\equiv$ which is in fact **equal to \mathcal{M} up to isomorphism**.

Notice also that, due to the good decidability properties of Muller tree automata, we can decide whether a given automaton $\mathcal{A}_=$ accepts an equivalence relation E_\equiv on $L(\mathcal{A})$ and whether, for each $i \in [1, n]$, the automaton \mathcal{A}_i accepts an n_i -ary relation R'_i on $L(\mathcal{A})$ such that E_\equiv is compatible with R'_i .

We get the definition of ω -automatic (injective) presentation of a structure and of ω -automatic structure by replacing simply Muller tree automata by Büchi automata in the above definition.

We recall now two important properties of automatic structures.

Theorem 2.6 (see [Blu99]) *The class of ω -tree-automatic (respectively, ω -automatic) structures is closed under first-order interpretations. In other words if \mathcal{M} is an ω -tree-automatic (respectively, ω -automatic) structure and \mathcal{M}' is a relational structure which is first-order interpretable in the structure \mathcal{M} , then the structure \mathcal{M}' is also ω -tree-automatic (respectively, ω -automatic).*

Theorem 2.7 (see [Hod83, Blu99]) *The first-order theory of an ω -tree-automatic (respectively, ω -automatic) structure is decidable.*

Notice that ω -(tree)-automatic structures are always *relational* structures. However we can also consider structures equipped with functional operations like groups, by replacing as usually a n -ary function by its graph which is a $(n + 1)$ -ary relation. This will be always the case in the sequel where all structures are viewed as relational structures.

Some examples of ω -automatic structures can be found in [Rub04, Nie07, KNRS07, KR03, BG04, KL08, HKMN08].

A first one is simply the boolean algebra of subsets of \mathbb{N} . The boolean algebra $\mathcal{P}(\mathbb{N})$ has an injective ω -automatic presentation where any subset $P \subseteq \mathbb{N}$ is simply represented by an infinite word x_P over the alphabet $\{0, 1\}$ defined by $x_P(i) = 1$ iff $i - 1 \in P$ for all integers $i \in \mathbb{N}$. It is easy to see that the inclusion relation is then definable by a Büchi automaton, as well as the operations of union, intersection, and complementation.

The additive group $(\mathbb{R}, +)$ is ω -automatic, as is the product $(\mathbb{R}, +) \times (\mathbb{R}, +)$.

Assume that a finite alphabet Σ is linearly ordered. Then the set $(\Sigma^\omega, \leq_{lex})$ of infinite words over the alphabet Σ , equipped with the lexicographic ordering, is also ω -automatic.

It is easy to see that every (injectively) ω -automatic structure is also (injectively) ω -tree-automatic. Indeed a Muller tree automaton can easily simulate

a Büchi automaton on the leftmost branch of an infinite tree. The converse is not true, as shown in [HKMN08] with the following example.

Let $\mathbb{T} = \{l, r\}^*$. An element of \mathbb{T} is simply a finite word over the alphabet $\{l, r\}$. A subset U of \mathbb{T} can be identified with an infinite binary tree $t_U \in T_{\{0,1\}}^\omega$ such that for all $u \in \{l, r\}^*$ it holds that $t_U(u) = 1$ if and only if $u \in U$. Let then $V = \{B \subseteq \mathbb{T} \mid t_B \notin \text{Path}((0^*.1)^\omega)\}$, where we denote $\text{Path}((0^*.1)^\omega)$ the set of infinite trees t in $T_{\{0,1\}}^\omega$ such that t has (at least) one path in $(0^*.1)^\omega$. The structure $(\mathcal{P}(\mathbb{T}), \subseteq, V)$ is ω -tree-automatic but not ω -automatic, see [HKMN08].

3 Two ω -tree automatic boolean algebras

We introduce in this section two ω -tree-automatic boolean algebras \mathcal{B}_1 and \mathcal{B}_2 and we will show later in Section 6 that the statement “ \mathcal{B}_1 is isomorphic to \mathcal{B}_2 ” is independent from the axiomatic system ZFC.

Some examples of ω -automatic, hence also ω -tree-automatic, structures have been already given above. Another example is the boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ of subsets of \mathbb{N} modulo *finite sets*. The set Fin of finite subsets of \mathbb{N} is an ideal of $\mathcal{P}(\mathbb{N})$, i.e. a subset of the powerset of \mathbb{N} such that:

1. $\emptyset \in \text{Fin}$ and $\mathbb{N} \notin \text{Fin}$.
2. For all $B, B' \in \text{Fin}$, it holds that $B \cup B' \in \text{Fin}$.
3. For all $B, B' \in \mathcal{P}(\mathbb{N})$, if $B \subseteq B'$ and $B' \in \text{Fin}$ then $B \in \text{Fin}$.

For any two subsets A and B of \mathbb{N} we denote $A \Delta B$ their symmetric difference. Then the relation \approx defined by: “ $A \approx B$ iff the symmetric difference $A \Delta B$ is finite” is an equivalence relation on $\mathcal{P}(\mathbb{N})$. The quotient $\mathcal{P}(\mathbb{N})/\approx$ denoted $\mathcal{P}(\mathbb{N})/\text{Fin}$ is a boolean algebra. It is easy to see that this boolean algebra is ω -automatic. We denote $[A]$ the equivalence class of a set of integers A . Let $\Sigma = \{0, 1\}$ and $L(\mathcal{A}) = \Sigma^\omega$ and for any $x \in \Sigma^\omega$, $h(x) = [\{i \in \mathbb{N} \mid x(i+1) = 1\}]$. Then it is easy to see that $\{(u, v) \in (\Sigma^\omega)^2 \mid h(u) = h(v)\}$ is accepted by a Büchi automaton. Similarly the relation $\{(u, v) \in (\Sigma^\omega)^2 \mid h(u) \subseteq^* h(v)\}$ is a regular ω -language because the “almost inclusion” relation \subseteq^* satisfies that $h(u) \subseteq^* h(v)$ iff $u(i) \leq v(i)$ for almost all integers i .

The operations \cap, \cup, \neg , of intersection, union, and complementation, on $\mathcal{P}(\mathbb{N})/\text{Fin}$ are defined by: $[B] \cap [B'] = [B \cap B']$, $[B] \cup [B'] = [B \cup B']$, and $\neg[B] = [\neg B]$, see [Jec02].

The operations of intersection, union, (respectively, complementation), considered as ternary relations (respectively, binary relation) are also given by regular ω -languages.

$\mathbf{0} = [\emptyset]$ is the equivalence class of the empty set and $\mathbf{1} = [\mathbb{N}]$ is the class of \mathbb{N} .

The structures $(\mathcal{P}(\mathbb{N})/Fin, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ and $(\mathcal{P}(\mathbb{N})/Fin, \subseteq^*)$ are ω -automatic, hence also ω -tree-automatic.

We are going to consider now another boolean algebra. Let $\mathbb{T} = \{l, r\}^*$ be the set of finite words over the alphabet $\{l, r\}$. A subset B of \mathbb{T} has no infinite antichain (for the prefix order) iff there is no infinite subset D of B such that for all $u, v \in D$, with $u \neq v$, u and v are incomparable for the prefix order relation \sqsubseteq .

Let then $I = \{B \subseteq \{l, r\}^* \mid B \text{ has no infinite antichain}\}$. The set I is an ideal of $\mathcal{P}(\mathbb{T})$, i.e. it is a subset of the powerset of \mathbb{T} such that:

1. $\emptyset \in I$ and $\mathbb{T} \notin I$.
2. For all $B, B' \in I$, it holds that $B \cup B' \in I$.
3. For all $B, B' \in \mathcal{P}(\mathbb{T})$, if $B \subseteq B'$ and $B' \in I$ then $B \in I$.

We can now consider the quotient $\mathcal{P}(\mathbb{T})/I$ of the set of subsets of \mathbb{T} modulo the ideal I . The relation \approx_I defined on $\mathcal{P}(\mathbb{T})$ by: “ $A \approx_I B$ iff the symmetric difference $A \Delta B$ is in I ” is an equivalence relation on $\mathcal{P}(\mathbb{T})$. The quotient $\mathcal{P}(\mathbb{T})/I$ is a boolean algebra.

We are going to show that this boolean algebra is actually ω -tree-automatic. Recall that a subset B of \mathbb{T} can be identified with an infinite binary tree $t_B \in T_{\{0,1\}}^\omega$ such that for all $u \in \{l, r\}^*$ it holds that $t_B(u) = 1$ if and only if $u \in B$. We can now state the following easy lemma.

Lemma 3.1 *Let $B \subseteq \{l, r\}^*$. Then the set B has an infinite antichain if and only if the tree t_B has an infinite branch, whose nodes form a sequence $\varepsilon = u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \dots$ and there exist infinitely many integers n_i such that:*
 $u_{n_i}.a \sqsubseteq v_i$ for some $a \in \{l, r\}$ and $v_i \in B$, and
 $u_{n_i+1} = u_{n_i}.b$ for $b \in \{l, r\}$ and $b \neq a$.

Proof. Assume first that a tree t_B has an infinite branch, whose nodes form a sequence $\varepsilon = u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \dots$ and there exist infinitely many integers n_i such that:

$u_{n_i}.a \sqsubseteq v_i$ for some $a \in \{l, r\}$ and $v_i \in B$, and

$u_{n_i+1} = u_{n_i}.b$ for $b \in \{l, r\}$ and $b \neq a$.

Then it is easy to see that the nodes $v_i \in B$ form an infinite antichain of B .

Conversely, assume that $B \subseteq \{l, r\}^*$ has an infinite antichain formed by nodes $w_i, i \geq 1$. We can easily construct by induction an infinite sequence of nodes $\varepsilon = u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \dots$ forming an infinite branch b and such that for each integer $i \geq 1$ there are infinitely many nodes w_j such that $u_i \sqsubseteq w_j$. Assume now, towards a contradiction, that there are only finitely many integers n_i such that:

$u_{n_i}.a \sqsubseteq w_{l_i}$ for some $a \in \{l, r\}$ and $l_i \geq 1$, and

$u_{n_i+1} = u_{n_i}.b$ for $b \in \{l, r\}$ and $b \neq a$.

Then there exists an integer N which is greater than all these integers n_i . Consider the node u_N of the branch b . By construction there are infinitely many integers j such that $u_N \sqsubseteq w_j$. But all these nodes w_j should be on the branch b . This would imply that the nodes $w_i, i \geq 1$, do not form an infinite antichain. Thus this would lead to a contradiction so there are *infinitely many* integers n_i such that:

$u_{n_i}.a \sqsubseteq w_{l_i}$ for some $a \in \{l, r\}$ and $l_i \geq 1$, and

$u_{n_i+1} = u_{n_i}.b$ for $b \in \{l, r\}$ and $b \neq a$. □

We can now state the following result.

Lemma 3.2 *The set $T = \{t_B \mid B \subseteq \{l, r\}^* \text{ and } B \text{ has an infinite antichain}\}$ is a regular tree language.*

Proof. We can construct a Muller tree automaton \mathcal{A} accepting the set T . We explain informally the behaviour of this automaton. Using the non-determinism the automaton, when reading a tree t_B , will guess an infinite branch whose existence is given by the preceding lemma, and integers n_i having the property given by the same lemma. □

We know that the class of regular tree languages is effectively closed under complementation. So we get now the following result.

Lemma 3.3 *The set $T_I = \{t_B \mid B \subseteq \{l, r\}^* \text{ and } B \text{ has no infinite antichain}\}$ is a regular tree language.*

Recall that, if we denote by $[A]$ the equivalence class of a set $A \subseteq \mathbb{T} = \{l, r\}^*$, then the operations of intersection, union, and complementation, on $\mathcal{P}(\mathbb{T})/I$ are defined by: $[B] \cap [B'] = [B \cap B']$, $[B] \cup [B'] = [B \cup B']$, and $\neg[B] = [\neg B]$. The almost inclusion relation \subseteq^* is defined by: $[B] \subseteq^* [B']$ iff $B \setminus B' \in I$.

We can now state the following result which will be fundamental in the sequel.

Proposition 3.4 *The boolean algebra $(\mathcal{P}(\mathbb{T})/I, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ and the structure $(\mathcal{P}(\mathbb{T})/I, \subseteq^*)$ are ω -tree-automatic structures.*

Proof. We denote by $[A]$ the equivalence class of a set $A \subseteq \mathbb{T} = \{l, r\}^*$. The class $[A]$ will be represented by the trees $t_B \in T_{\{0,1\}}^\omega$ such that $[A] = [B]$, i.e. such that the symmetric difference $A \Delta B$ is in I . The function h will associate the class $[B]$ to each tree $t_B \in T_{\{0,1\}}^\omega$. Then it is easy to see, from Lemma 3.3, that $\{(t, t') \in T_{\{0,1\}}^\omega \times T_{\{0,1\}}^\omega \mid h(t) = h(t')\}$ is accepted by a Muller tree automaton accepting infinite trees in $T_{\{0,1\} \times \{0,1\}}^\omega$.

From Lemma 3.3, we can also easily infer that the almost inclusion relation and the operations of intersection, union, and complementation, are also given by regular tree languages. $\mathbf{0} = [\emptyset]$ is simply represented by the trees in the set T_I . The class $\mathbf{1} = [\mathbb{T}]$ is represented by the trees t_B , where $B \subseteq \{l, r\}^*$ and $\neg B$ has no infinite antichain. \square

From now on we shall denote $\mathcal{B}_1 = (\mathcal{P}(\mathbb{N})/Fin, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ and $\mathcal{B}_2 = (\mathcal{P}(\mathbb{T})/I, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ the two ω -tree-automatic boolean algebras defined above.

Recall now the definition of an atomless boolean algebra.

Definition 3.5 *Let $\mathcal{B} = (B, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ be a boolean algebra and \subseteq be the inclusion relation on B defined by $x \subseteq y$ iff $x \cap y = x$ for all $x, y \in B$. Then the boolean algebra \mathcal{B} is said to be an atomless boolean algebra iff for every $x \in B$ such that $x \neq \mathbf{0}$ there exists a $z \in B$ such that $\mathbf{0} \subset z \subset x$.*

We can now state the following result.

Proposition 3.6 *The two boolean algebras \mathcal{B}_1 and \mathcal{B}_2 are atomless boolean algebras.*

Proof. Consider firstly the boolean algebra $\mathcal{B}_1 = (\mathcal{P}(\mathbb{N})/Fin, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$. It is well known that it is an atomless boolean algebra. We now give a proof of this result. Let $A \subseteq \mathbb{N}$ be such that the equivalence class $[A]$ is different from the element $\mathbf{0}$ in \mathcal{B}_1 . Then the set A is infinite and there exist two infinite sets A_1 and A_2 such that $A = A_1 \cup A_2$. The element $[A_1]$ is different from the

element $\mathbf{0}$ in \mathcal{B}_1 because A_1 is infinite, and $[A_1] \subset [A]$ because $A - A_1 = A_2$ is infinite. Thus the following strict inclusions hold in \mathcal{B}_1 : $\mathbf{0} \subset [A_1] \subset [A]$.

We can prove in a similar way that the boolean algebra \mathcal{B}_2 is an atomless boolean algebra. Let then $X \subseteq \mathbb{T}$ be such that the equivalence class $[X]$ is different from the element $\mathbf{0}$ in \mathcal{B}_2 . Then the set $X \subseteq \{l, r\}^*$ contains an infinite antichain $A \subseteq X$. There are two infinite sets A_1 and A_2 such that $A = A_1 \cup A_2$. These two sets are also infinite antichains, so $[A_1]$ is different from $\mathbf{0}$ in \mathcal{B}_2 and $[A_1] \subset [X]$ because the set $X - A_1$ contains the infinite antichain A_2 . Thus the following strict inclusions hold in \mathcal{B}_2 : $\mathbf{0} \subset [A_1] \subset [X]$. This proves that the boolean algebra \mathcal{B}_2 is atomless. \square

4 Topology

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80, LT94, Kec95, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\text{pref}(u,v)}}$ where $l_{\text{pref}(u,v)}$ is the first integer n such that $u(n+1) \neq v(n+1)$. This metric induces on Σ^ω the usual Cantor topology for which *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^*$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set. We recall now a characterization of closed sets which will be useful in the sequel.

Proposition 4.1 (see [Sta97, Kec95]) *A set $L \subseteq \Sigma^\omega$ is a closed subset of Σ^ω iff for every $\sigma \in \Sigma^\omega$,*

$[\forall n \geq 1, \exists u \in \Sigma^\omega \text{ such that } \sigma(1) \dots \sigma(n).u \in L]$ implies that $\sigma \in L$.

We now define the next classes of the Borel Hierarchy of subsets of Σ^ω .

Definition 4.2 *For an integer $n \geq 1$, the classes Σ_n^0 and Π_n^0 of the Borel Hierarchy on the topological space Σ^ω are defined as follows:*

Σ_1^0 is the class of open subsets of Σ^ω , Π_1^0 is the class of closed subsets of Σ^ω , and for any integer $n \geq 1$:

Σ_{n+1}^0 is the class of countable unions of Π_n^0 -subsets of Σ^ω .

Π_{n+1}^0 is the class of countable intersections of Σ_n^0 -subsets of Σ^ω .

Notation 4.3 *Following the earlier notations for the Borel hierarchy of Borel sets of finite rank, a Π_2^0 -set is also called a G_δ -set and a Σ_2^0 -set is also called a F_σ -set. So a G_δ -set is a countable intersection of open sets and a F_σ -set is a countable union of closed sets.*

The Borel Hierarchy is also defined for transfinite levels indexed by countable ordinals, see [Mos80, Kec95]. However we shall not need these notions in the sequel. Recall that the class of Borel subsets of Σ^ω is the closure of the class of open subsets of Σ^ω under countable union and countable intersection.

There exists another hierarchy beyond the Borel hierarchy, which is called the projective hierarchy. The first level of the projective hierarchy is formed by the class Σ_1^1 of *analytic sets* and the class Π_1^1 of *co-analytic sets* which are complements of analytic sets. In particular the class of Borel subsets of Σ^ω is strictly included in the class Σ_1^1 of *analytic sets* which are obtained by projection of Borel sets.

Definition 4.4 *A subset A of Σ^ω is in the class Σ_1^1 of **analytic** sets iff there exists a finite set Y and a Borel subset B of $(\Sigma \times Y)^\omega$ such that $x \in A \leftrightarrow [\exists y \in Y^\omega \ (x, y) \in B]$, where (x, y) is the infinite word over the alphabet $\Sigma \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.*

An important fact in this paper is that the powerset $\mathcal{P}(\mathbb{N})$ can be equipped with the standard metric topology obtained from its identification with the Cantor space $\{0, 1\}^\omega$. Then the topological notions like open, closed, F_σ , analytic, can be applied to families of subsets of \mathbb{N} .

The ideal Fin of $\mathcal{P}(\mathbb{N})$ is identified with the set of ω -words over the alphabet $\{0, 1\}$ having only finitely many letters 1. It is a well known example of F_σ -subset of $\{0, 1\}^\omega$, as stated in the following lemma.

Lemma 4.5 *[see [PP04, Kec95]] The powerset $\mathcal{P}(\mathbb{N})$ being identified with the Cantor space $\{0, 1\}^\omega$, the ideal Fin of $\mathcal{P}(\mathbb{N})$ is a F_σ -subset of $\{0, 1\}^\omega$.*

Proof. Let $k \geq 1$ be an integer and Fin_k be the set of subsets of \mathbb{N} having at most k elements. The set Fin_k is identified with the set of ω -words over the alphabet $\{0, 1\}$ having at most k letters 1. Using Proposition 4.1 it is easy to see that for each $k \geq 1$ the set Fin_k is then a closed subset of $\{0, 1\}^\omega$. This follows from the fact that if a set $A \subseteq \mathbb{N}$ is such that every finite subset of A has at most k elements, then the set A itself has at most k elements. Thus $Fin = \bigcup_{k \geq 1} Fin_k$ is a countable union of closed sets, i.e. a F_σ -subset of $\{0, 1\}^\omega$. \square

Consider now the set $\mathbb{T} = \{l, r\}^*$ of finite words over the alphabet $\{l, r\}$. This set is countably infinite and we can define a bijection from $\mathbb{T} = \{l, r\}^*$ onto \mathbb{N} by enumerating the elements of \mathbb{T} . For each integer $n \geq 0$, call W_n the set of

words of length n of $\{l, r\}^*$. Then $W_0 = \{\varepsilon\}$, $W_1 = \{l, r\}$, $W_2 = \{ll, lr, rl, rr\}$ and so on. W_n is the set of nodes which appear in the $(n+1)^{th}$ level of an infinite binary tree. We consider now the lexicographic order on W_n (assuming that l is before r for this order). Then, in the enumeration of the nodes with regard to this order, the nodes of W_1 will be: l, r ; the nodes of W_2 will be: ll, lr, rl, rr ; the nodes of W_3 will be: $lll, llr, lrl, lrr, rll, rlr, rrl, rrr$. We enumerate now the elements of \mathbb{T} in the following order. We begin with ε , then the nodes in W_1 in the lexicographic order, then the nodes in W_2 in the lexicographic order, then the nodes in W_3 in the lexicographic order, and so on ... The successive nodes are then

$$\varepsilon, l, r, ll, lr, rl, rr, lll, llr, lrl, lrr, rll, rlr, rrl, rrr, \dots$$

For every $u \in \{l, r\}^*$, we define $f(u) \in \mathbb{N}$ such that u is the $(f(u) + 1)^{th}$ element in the above enumeration of words of $\{l, r\}^*$. For instance $f(\varepsilon) = 0$, $f(l) = 1$, $f(r) = 2$, $f(ll) = 3$, $f(lr) = 4$, and so on ...

The function f is then a bijection from $\{l, r\}^*$ onto \mathbb{N} , and it induces also a bijection from $\mathcal{P}(\mathbb{T})$ onto $\mathcal{P}(\mathbb{N})$ which will be also denoted by f , the meaning being clear from the context.

Recall that we have set $I = \{B \subseteq \{l, r\}^* \mid B \text{ has no infinite antichain}\}$, and that the set I is an ideal of $\mathcal{P}(\mathbb{T})$. We are going to show that $f(I)$ is a F_σ -subset of $\{0, 1\}^\omega$, where again the powerset $\mathcal{P}(\mathbb{N})$ is identified with the Cantor space $\{0, 1\}^\omega$. We first state the following lemma.

Lemma 4.6 *Let $I = \{B \subseteq \{l, r\}^* \mid B \text{ has no infinite antichain}\}$ and, for each integer $k \geq 1$, $I_k = \{B \subseteq \{l, r\}^* \mid B \text{ has no antichain of cardinal greater than } k\}$. Then $I = \bigcup_{k \geq 1} I_k$.*

Proof. It is clear that if $B \subseteq \{l, r\}^*$ has no antichain of cardinal greater than k , for an integer $k \geq 1$, then B has no infinite antichain. Then the inclusion $\bigcup_{k \geq 1} I_k \subseteq I$ holds.

We want now to prove that $I \subseteq \bigcup_{k \geq 1} I_k$. We assume that a set $B \subseteq \{l, r\}^*$ is not in $\bigcup_{k \geq 1} I_k$ and we are going to prove that this implies that $B \notin I$.

Let then $B \subseteq \{l, r\}^*$ such that $B \notin \bigcup_{k \geq 1} I_k$. For every $k \geq 1$ it holds that $B \notin I_k$, i.e. B has some antichain of cardinal greater than k . We are going to prove that B has an infinite antichain, using the characterization given by Lemma 3.1. We can first construct by induction an infinite sequence of nodes of the tree t_B :

$$\varepsilon = u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \dots$$

such that for every integer $j \geq 0$ and every $k \geq 1$, there is an antichain in B of cardinal greater than k whose nodes have u_j as prefix. Indeed assume that we have already construct $\varepsilon = u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \dots \sqsubseteq u_j$. Then at least one node among $u_j.l$ and $u_j.r$ has the desired property and we can choose this node as u_{j+1} .

We can now see that there exist infinitely many integers n_i such that:

$$\begin{aligned} u_{n_i}.a &\sqsubseteq v_i \quad \text{for some } a \in \{l, r\} \text{ and } v_i \in B, \text{ and} \\ u_{n_i+1} &= u_{n_i}.b \quad \text{for } b \in \{l, r\} \text{ and } b \neq a. \end{aligned}$$

Indeed we can construct the sequence $(n_i)_{i \geq 0}$ by induction. Firstly there is an antichain in B of cardinal greater than 2 whose nodes have u_0 as prefix. Thus there is an integer $n_0 \geq 0$ such that: $u_{n_0}.a \sqsubseteq v_0$ for some $a \in \{l, r\}$ and $v_0 \in B$, and $u_{n_0+1} = u_{n_0}.b$ for $b \in \{l, r\}$ and $b \neq a$. Assume now that we have constructed integers n_0, n_1, \dots, n_j having the desired property. Then by construction of the sequence of nodes $(u_i)_{i \geq 0}$, we know that there is an antichain in B of cardinal greater than 2 whose nodes have u_{n_j+1} as prefix. Thus there is an integer $n_{j+1} \geq n_j + 1$ such that: $u_{n_{j+1}}.a \sqsubseteq v_{j+1}$ for some $a \in \{l, r\}$ and $v_{j+1} \in B$, and $u_{n_{j+1}+1} = u_{n_{j+1}}.b$ for $b \in \{l, r\}$ and $b \neq a$.

Using Lemma 3.1 we can conclude that B has an infinite antichain, i.e. $B \notin I$. This proves the inclusion $I \subseteq \bigcup_{k \geq 1} I_k$. \square

Lemma 4.7 *For each integer $k \geq 1$, the set $f(I_k)$ is a closed subset of the Cantor space $\{0, 1\}^\omega$.*

Proof. Let $k \geq 1$ be an integer. We shall prove that $f(I_k)$ is a closed subset of the Cantor space $\{0, 1\}^\omega$, using the characterization of closed sets given by Proposition 4.1.

Let $x \in \{0, 1\}^\omega$ such that $[\forall n \geq 1, \exists y_n \in \{0, 1\}^\omega \text{ such that } x(1) \dots x(n).y_n \in f(I_k)]$. This implies that if P_n is the subset of \mathbb{N} which is identified with the ω -word $x(1) \dots x(n).y_n$ then $f^{-1}(P_n) \subseteq \{l, r\}^*$ has no antichain of cardinal greater than k . In particular, for every $n \geq 1$, the finite set $f^{-1}(P_n \cap \{0, \dots, n-1\})$ has no antichain of cardinal greater than k . Thus every finite subset of $f^{-1}(x)$ has no antichain of cardinal greater than k . This implies that $f^{-1}(x)$ itself has no antichain of cardinal greater than k , i.e. $f^{-1}(x)$ is in I_k and x belongs to $f(I_k)$.

Using Proposition 4.1 we can conclude that $f(I_k)$ is a closed subset of the Cantor space $\{0, 1\}^\omega$. \square

We can now state the following result, which will be important in the sequel.

Proposition 4.8 *Let $I = \{B \subseteq \{l, r\}^* \mid B \text{ has no infinite antichain}\}$. Then the set $f(I)$ is a F_σ -subset of $\{0, 1\}^\omega$.*

Proof. By Lemma 4.6, it holds that $I = \bigcup_{k \geq 1} I_k$. But by Lemma 4.7 for each integer $k \geq 1$, the set $f(I_k)$ is a closed subset of the Cantor space $\{0, 1\}^\omega$. Thus $f(I) = \bigcup_{k \geq 1} f(I_k)$ is a countable union of closed sets, i.e. a F_σ -subset of $\{0, 1\}^\omega$. \square

5 Axioms of set theory

We now recall some basic notions of set theory which will be useful in the sequel, and which are exposed in a textbook on set theory, like [Jec02].

The usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. A model (\mathbf{V}, \in) of the axiomatic system ZFC is a collection \mathbf{V} of sets, equipped with the membership relation \in , where “ $x \in y$ ” means that the set x is an element of the set y , which satisfies the axioms of ZFC. We shall often say “the model \mathbf{V} ” instead of “the model (\mathbf{V}, \in) ”.

We recall that the infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \dots, \aleph_\alpha, \dots$ and that Cantor’s Continuum Hypothesis CH states that the cardinality of the continuum 2^{\aleph_0} is equal to the first uncountable cardinal \aleph_1 .

Recall also that OCA denotes the Open Coloring Axiom, a natural alternative to CH which has been first considered by the second author in [Tod89].

For any set X we denote $[X]^2$ the set of subsets of X having exactly 2 elements. Let \mathbb{R} be the set of reals equipped with the usual topology. If $X \subseteq \mathbb{R}$ and $K \subseteq [X]^2$ then we say that K is open if $\{(x, y) \mid \{x, y\} \in K\}$ is open in $X \times X$.

The OCA states that if $X \subseteq \mathbb{R}$ and $[X]^2 = K_0 \cup K_1$ is a partition of $[X]^2$ with K_0 open, then either there exists an uncountable subset Y of X such that $[Y]^2 \subseteq K_0$ or there exist a sequence of sets $(H_n)_{n \in \omega}$, such that $X = \bigcup_{n \in \omega} H_n$ and, for all $n \in \omega$, $[H_n]^2 \subseteq K_1$.

The above version of the axiom OCA can be shown to be equivalent to the following one:

If $G = (V, E)$ is a graph whose edge relation E can be written as a countable union of 'rectangles', i.e., sets of the form

$$\{\{x, y\} : x \in P, y \in Q\}$$

for $P, Q \subseteq V$, then either the chromatic number of G is countable, or else G has an uncountable clique.

It is known that if the theory ZFC is consistent, then so are the theories (ZFC + CH) and (ZFC + OCA), see [Jec02, pages 176 and 577].

We now recall some known properties of the class \mathbf{L} of *constructible sets* in a model \mathbf{V} of ZF, which will be useful in the sequel. If \mathbf{V} is a model of ZF and \mathbf{L} is the class of *constructible sets* of \mathbf{V} , then the class \mathbf{L} forms a model of (ZFC + CH). Notice that the axiom ($\mathbf{V}=\mathbf{L}$) means "every set is constructible" and that it is consistent with ZFC.

In particular, if \mathbf{V} is a model of (ZFC + OCA) and if \mathbf{L} is the class of *constructible sets* of \mathbf{V} , then the class \mathbf{L} forms a model of (ZFC + CH).

6 The isomorphism relation

We are going to see that the statement " \mathcal{B}_1 is isomorphic to \mathcal{B}_2 " is independent from the axiomatic system ZFC.

Theorem 6.1

1. (ZFC + CH) \mathcal{B}_1 is isomorphic to \mathcal{B}_2 .
2. (ZFC + OCA) \mathcal{B}_1 is not isomorphic to \mathcal{B}_2 .

Proof. We have already seen that the powerset $\mathcal{P}(\mathbb{N})$ can be equipped with the standard metric topology obtained from its identification with the Cantor space $\{0, 1\}^\omega$.

Then the ideal Fin of $\mathcal{P}(\mathbb{N})$ is identified with the set of ω -words over the alphabet $\{0, 1\}$ having only finitely many letters 1 and Lemma 4.5 states that it is a F_σ -subset of $\{0, 1\}^\omega$. Thus the boolean algebra \mathcal{B}_1 is a quotient algebra of $\mathcal{P}(\mathbb{N})$ over a F_σ -ideal.

Consider now the boolean algebra $\mathcal{B}_2 = \mathcal{P}(\mathbb{T})/I$. Recall that we have defined a bijection f from $\mathbb{T} = \{l, r\}^*$ onto \mathbb{N} by enumerating the elements of \mathbb{T} . Then

the boolean algebra \mathcal{B}_2 is isomorphic to the boolean algebra $\mathcal{P}(\mathbb{N})/f(I)$. But by Proposition 4.8, the set $\mathcal{P}(\mathbb{N})$ being again identified with the Cantor set $\{0,1\}^\omega$, the set $f(I)$ is a F_σ -subset of $\{0,1\}^\omega$. Thus the boolean algebra \mathcal{B}_2 is also isomorphic to a quotient algebra of $\mathcal{P}(\mathbb{N})$ over a F_σ -ideal.

On the other hand, Just and Krawczyk proved in [JK84, Theorem 1] that two boolean algebras which are quotients of $\mathcal{P}(\mathbb{N})$ over F_σ -ideals are always isomorphic under (ZFC + CH). This implies the first part of the Theorem.

On the other hand, it is proved in [Tod98, Theorem 6](see also [Far00] and [Jus92] for more information about the influence of OCA to the quotient structures of this sort) that under OCA if a quotient algebra $\mathcal{P}(\mathbb{N})/J$ over an analytic ideal J on \mathbb{N} is isomorphic to a subalgebra of $\mathcal{P}(\mathbb{N})/Fin$ then J must be a *trivial modification* of the ideal Fin , i.e., there is some infinite subset B of \mathbb{N} such that

$$J = \{A \subseteq \mathbb{N} : A \cap B \in Fin\}.$$

Consider now $I = \{B \subseteq \{l, r\}^* \mid B \text{ has no infinite antichain}\}$ and the ideal $J = f(I)$. By Proposition 4.8 the ideal $f(I)$ is a F_σ hence also analytic subset of $\{0,1\}^\omega$.

Let us now prove that the ideal $J = f(I)$ is not a trivial modification of Fin . Towards a contradiction, assume on the contrary that there exists a subset B of \mathbb{N} such that $J = \{A \subseteq \mathbb{N} : A \cap B \in Fin\}$. Notice that if $C \subseteq \{l, r\}^*$ is a (finite or infinite) chain, i.e. is linearly ordered by the prefix order relation, then it has no infinite antichain and it belongs to I . In particular, for every integer $n \geq 1$ the set $C_n = \{l^n.r^k \mid k \geq 1\}$ is in I so $f(C_n)$ is in $J = f(I)$ and $f(C_n) \cap B$ would be finite. This would imply that for every integer $n \geq 1$ there would exist an integer $k_n \geq 1$ such that $f(\{l^n.r^k \mid k \geq k_n\}) \cap B$ is empty. Let now $A = \{l^n.r^{k_n} \mid n \geq 1\}$. It is an infinite antichain so it is not in I . Thus $f(A)$ is not in J and $f(A) \cap B$ should be infinite. But by construction $f(A) \cap B$ is empty and this leads to a contradiction. This proves that the ideal $J = f(I)$ is not a trivial modification of Fin .

Thus, assuming OCA, the boolean algebra $\mathcal{P}(\mathbb{N})/f(I)$ is not even isomorphic to a subalgebra of \mathcal{B}_1 . But the boolean algebra \mathcal{B}_2 is isomorphic to the boolean algebra $\mathcal{P}(\mathbb{N})/f(I)$ therefore the boolean algebra \mathcal{B}_2 is not even isomorphic to a subalgebra of \mathcal{B}_1 . \square

We can now state the following result.

Corollary 6.2 *The isomorphism relation for ω -tree-automatic structures (respectively, ω -tree-automatic boolean algebras, ω -tree-automatic partial orders) is not determined by the axiomatic system ZFC.*

Proof. The result for ω -tree-automatic boolean algebras, hence also for ω -tree-automatic structures, follows directly from Theorem 6.1 and the fact that the boolean algebras \mathcal{B}_1 and \mathcal{B}_2 are ω -tree-automatic. For partial orders, we consider the ω -tree-automatic structures $(\mathcal{P}(\mathbb{N})/Fin, \subseteq^*)$ and $(\mathcal{P}(\mathbb{T})/I, \subseteq^*)$. These two structures are isomorphic if and only if the two boolean algebras \mathcal{B}_1 and \mathcal{B}_2 are isomorphic, see [Jec02, page 79]. Then the result for partial orders follows from the case of boolean algebras. \square

We are going to get similar results for other classes of ω -tree-automatic structures. First we can consider a boolean algebra $(B, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ as a commutative ring with unity element $(B, \Delta, \cap, \mathbf{1})$, where Δ is the symmetric difference operation. It is clear that the operations of union and complementation can be defined from the symmetric difference and intersection operations. Moreover two boolean algebras $(B, \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ and $(B', \cap, \cup, \neg, \mathbf{0}, \mathbf{1})$ are isomorphic if and only if the rings $(B, \Delta, \cap, \mathbf{1})$ and $(B', \Delta, \cap, \mathbf{1})$ are isomorphic. We shall denote $\mathcal{R}_1 = (\mathcal{P}(\mathbb{N})/Fin, \Delta, \cap, \mathbf{1})$ and $\mathcal{R}_2 = (\mathcal{P}(\mathbb{T})/I, \Delta, \cap, \mathbf{1})$ the two commutative rings associated with the two boolean algebras \mathcal{B}_1 and \mathcal{B}_2 . Notice that $[\emptyset]$ is the unity element for the operation Δ and that every element of the ring \mathcal{R}_1 (respectively, \mathcal{R}_2) is its own inverse for this group operation. On the other hand $\mathbf{1}$ is the unity element for the operation \cap in both rings and it is also the unique invertible element for the operation \cap in both rings.

We can now state the following result, which follows directly from Theorem 6.1.

Theorem 6.3

1. (ZFC + CH) \mathcal{R}_1 is isomorphic to \mathcal{R}_2 .
2. (ZFC + OCA) \mathcal{R}_1 is not isomorphic to \mathcal{R}_2 .

We can also obtain a similar result for non commutative rings. For that purpose we consider the set $M_n(R)$ of square matrices with n columns and n rows and coefficients in a given ring R . If $n \geq 2$ then the set $M_n(R)$, equipped with addition and multiplication of matrices, is a non commutative ring. The ring $M_n(R)$ is first-order interpretable in the ring R ; each matrix

M being represented by a unique n^2 -tuple of elements of R , the addition and multiplication of matrices are first order definable in R .

On the other hand, the class of ω -tree-automatic structures is closed under first order interpretations. Thus if R is an ω -tree-automatic ring then the ring of matrices $M_n(R)$ is also ω -tree-automatic. We now denote $\mathcal{M}_1 = M_n(\mathcal{R}_1)$ and $\mathcal{M}_2 = M_n(\mathcal{R}_2)$, for some fixed integer $n \geq 2$, and state the following result.

Theorem 6.4

1. (ZFC + CH) \mathcal{M}_1 is isomorphic to \mathcal{M}_2 .
2. (ZFC + OCA) \mathcal{M}_1 is not isomorphic to \mathcal{M}_2 .

Proof. It is clear that if \mathcal{R}_1 is isomorphic to \mathcal{R}_2 then the rings \mathcal{M}_1 and \mathcal{M}_2 are also isomorphic. Conversely, assume that $\Phi : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is an isomorphism of rings. Then, for $i \in \{1, 2\}$, consider the center \mathcal{C}_i of the ring \mathcal{M}_i . The set \mathcal{C}_i contains the matrices of \mathcal{M}_i which commute with every matrix of \mathcal{M}_i . It holds that the restriction of Φ to \mathcal{C}_1 is an isomorphism between \mathcal{C}_1 and \mathcal{C}_2 . But it is well known that the center C of a ring $M_n(R)$ is formed by matrices $M(u)$ which have a same element $u \in R$ on the diagonal and zeros elsewhere. Then the center C of $M_n(R)$ is a subring of the ring $M_n(R)$ which is isomorphic to the ring R . Thus “ \mathcal{C}_1 is isomorphic to \mathcal{C}_2 ” implies that “ \mathcal{R}_1 is isomorphic to \mathcal{R}_2 ”. We have then proved that \mathcal{M}_1 is isomorphic to \mathcal{M}_2 if and only if \mathcal{R}_1 is isomorphic to \mathcal{R}_2 . The result follows then from Theorem 6.3. \square

We look now for similar results for groups. If R is a ring with unity element, i.e. a unitary ring, then the set $GL_n(R)$ of invertible matrices of $M_n(R)$ is a group for the multiplication of matrices. It is first order interpretable in the ring R because it is the set of matrices $M \in M_n(R)$ such that the determinant $\det(M)$ of M is invertible in R , see [Nie07]. This implies that if R is an ω -tree-automatic unitary ring then the group $GL_n(R)$ is also ω -tree-automatic.

Another interesting group is the unitriangular group $UT_n(R)$ for some integer $n \geq 3$ and R a unitary ring. A matrix $M \in M_n(R)$ is in the group $UT_n(R)$ if and only if it is an upper triangular matrix which has only coefficients 1 on the diagonal, where 1 is the unity element for the second operation of R . The group $UT_n(R)$ is also first order interpretable in the ring R and it is a subgroup of the group $GL_n(R)$. We recall now the classical notion of

nilpotent group. The center of a group G is denoted $Z(G)$. A group G is said to be nilpotent of class 1 iff G is non-trivial and abelian. The group G is said to be nilpotent of class $c + 1$ if and only if the group $G/Z(G)$ is nilpotent of class c . The group $UT_n(R)$ is a classical example of nilpotent group of class $n - 1$, see [Bel94].

We denote $\mathcal{U}_{i,n} = UT_n(\mathcal{R}_i)$ for each $i \in \{1, 2\}$. We have seen that the groups $\mathcal{U}_{i,n}$ are first order interpretable in the ring \mathcal{R}_i . Thus the groups $\mathcal{U}_{i,n}$ are ω -tree-automatic. We can now state the following result.

Theorem 6.5

1. (ZFC + CH) For each integer $n \geq 3$, $\mathcal{U}_{1,n}$ is isomorphic to $\mathcal{U}_{2,n}$.
2. (ZFC + OCA) For each integer $n \geq 3$, $\mathcal{U}_{1,n}$ is not isomorphic to $\mathcal{U}_{2,n}$.

Proof. If \mathcal{R}_1 is isomorphic to \mathcal{R}_2 then it is clear that for each integer $n \geq 2$, $\mathcal{U}_{1,n}$ is isomorphic to $\mathcal{U}_{2,n}$. On the other hand, Belegradek proved in [Bel94] that if $UT_n(R)$ and $UT_n(S)$ are isomorphic, for some integer $n \geq 3$ and some commutative rings R and S , then the rings R and S are also isomorphic. Thus if for some integer $n \geq 3$ the groups $\mathcal{U}_{1,n}$ and $\mathcal{U}_{2,n}$ are isomorphic then \mathcal{R}_1 is isomorphic to \mathcal{R}_2 .

Then we have proved that, for each integer $n \geq 3$, \mathcal{R}_1 is isomorphic to \mathcal{R}_2 if and only if the groups $\mathcal{U}_{1,n}$ and $\mathcal{U}_{2,n}$ are isomorphic. The result follows then directly from Theorem 6.3. \square

Then we can now infer the following result.

Corollary 6.6 *The isomorphism relation for ω -tree-automatic commutative rings (respectively, non-commutative rings, groups, nilpotent groups of class $n \geq 2$) is not determined by the axiomatic system ZFC.*

Proof. The result follows from Theorems 6.3, 6.4, 6.5, and the fact that the commutative rings \mathcal{R}_i , the non-commutative rings \mathcal{M}_i , the groups $\mathcal{U}_{i,n}$, are all ω -tree-automatic structures. \square

Remark 6.7 In [HKMN08] the authors show that the isomorphism relation for ω -automatic structures is not determined by the axiomatic system ZF. They considered the two ω -automatic groups $(\mathbb{R}, +)$ and $(\mathbb{R}, +) \times (\mathbb{R}, +)$. Assuming the axiom of choice is satisfied, these two groups are isomorphic because they are both \mathbb{Q} -vectorial spaces of the same dimension 2^{\aleph_0} . But

in Shelah's model of ZF where every set of reals is Baire measurable the two groups are not isomorphic, see [HKMN08]. We have then proved here a stronger result in the case of ω -tree-automatic structures: the isomorphism relation is not determined by the axiomatic system ZFC (and not only ZF). Moreover we have proved our result not only for the class of all ω -tree-automatic structures and for the class of ω -tree-automatic groups, but also for the classes of ω -tree-automatic boolean algebras (respectively, commutative rings, non-commutative rings, nilpotent groups of class $n \geq 2$).

Remark 6.8 The two boolean algebras \mathcal{B}_1 and \mathcal{B}_2 are not isomorphic in every model of ZFC. On the other hand, they are always elementarily equivalent because they are two atomless boolean algebras and the first order theory of atomless boolean algebras is complete. In a similar way the rings \mathcal{R}_1 and \mathcal{R}_2 are always elementarily equivalent. This implies that the rings \mathcal{M}_1 and \mathcal{M}_2 (respectively, the groups $\mathcal{U}_{1,n}$ and $\mathcal{U}_{2,n}$, for some integer $n \geq 3$) are also always elementarily equivalent because the ring $M_n(R)$ (respectively, the group $U_n(R)$ for $n \geq 3$) is first order interpretable in the ring R without parameters and **uniformly in the ring R** , see [Bel94, page 20].

An ω -tree-automatic presentation of a structure is given by a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq n})$, where $L(\mathcal{A}) \subseteq T_\Sigma^\omega$, the automaton $\mathcal{A}_=$ accepts an equivalence relation E_\equiv on $L(\mathcal{A})$, and for each $i \in [1, n]$, the automaton \mathcal{A}_i accepts an n_i -ary relation R_i on $L(\mathcal{A})$ such that E_\equiv is compatible with R_i . Then the tuple of automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq n})$ gives an ω -tree-automatic presentation of the quotient structure $(L(\mathcal{A}), (R_i)_{1 \leq i \leq n})/E_\equiv$.

Notice that the tuple of automata can be coded by a finite sequence of symbols, hence by a unique integer N . If N is the code of the tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq n})$ we shall denote \mathcal{S}_N the ω -tree-automatic structure $(L(\mathcal{A}), (R_i)_{1 \leq i \leq n})/E_\equiv$.

The isomorphism problem for ω -tree-automatic structures is:

$$\{(n, m) \in \mathbb{N}^2 \mid \mathcal{S}_n \text{ is isomorphic to } \mathcal{S}_m\}.$$

A similar definition is given for automatic structures presentable by finite automata reading finite words, or for ω -automatic structures presentable by Büchi automata reading infinite words. It is proved in [KNRS07] that the isomorphism problem for automatic structures is Σ_1^1 -complete. The authors proved in [HKMN08] that the isomorphism problem for ω -automatic structures is not a Σ_2^1 -set. In fact their proof implies also that this isomorphism problem is not a Π_2^1 -set. Moreover this is also the case for the restricted class

of ω -automatic (abelian) groups and for the class of all ω -tree-automatic structures which is an extension of the class of ω -automatic structures.

We can now infer from above independence results some similar results for many other classes of ω -tree-automatic structures.

Theorem 6.9 *The isomorphism problem for ω -tree-automatic boolean algebras (respectively, partial orders, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $n \geq 2$) is neither a Σ_2^1 -set nor a Π_2^1 -set.*

Proof. We prove first the result for ω -tree-automatic boolean algebras. By Theorem 6.1 we know that if ZF hence also ZFC is consistent then there is a model \mathbf{V} of (ZFC + OCA) in which the boolean algebra \mathcal{B}_1 is not isomorphic to the boolean algebra \mathcal{B}_2 . But the inner model \mathbf{L} of constructible sets in \mathbf{V} is a model of (ZFC + CH) so in this model the two boolean algebras \mathcal{B}_1 and \mathcal{B}_2 are isomorphic. We have also proved that these two boolean algebras are ω -tree-automatic.

On the other hand, Schoenfield's Absoluteness Theorem implies that every Σ_2^1 -set (respectively, Π_2^1 -set) is absolute for all inner models of (ZF + DC), where (DC) is a weak version of the axiom of choice called axiom of dependent choice which holds in particular in the inner model \mathbf{L} , see [Jec02, page 490].

In particular, if the isomorphism problem for ω -tree-automatic boolean algebras was a Σ_2^1 -set (respectively, a Π_2^1 -set), then it could not be a different subset of \mathbb{N}^2 in the models \mathbf{V} and \mathbf{L} considered above. Thus the isomorphism problem for ω -tree-automatic boolean algebras is neither a Σ_2^1 -set nor a Π_2^1 -set.

The other cases of partial orders, rings, commutative rings, non commutative rings, non commutative groups, nilpotent groups of class $n \geq 2$, follow in the same way from Theorems 6.3, 6.4, and 6.5. \square

Remark 6.10 The set of codes of tuples of Muller tree automata which form ω -tree-automatic presentations of boolean algebras is recursive because the first-order theory of boolean algebras is finitely axiomatizable and the first-order theory of an ω -tree-automatic structure is decidable by Theorem 2.7. The same result holds in the cases of partial orders (respectively, rings, commutative rings, non commutative rings, non commutative groups). Moreover this is also the case for nilpotent groups of class $n \geq 2$. Indeed it is decidable

whether a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, \mathcal{A}_1)$ is an ω -tree-automatic presentation of an abelian or a non-abelian group $(G, +)$. If it is an ω -tree-automatic presentation of a non-abelian group $(G, +)$, then the center $Z(G)$ is first-order definable hence represented by a regular subset of $L(\mathcal{A})$, and we can get an ω -tree-automatic presentation of the quotient group $G/Z(G)$. If the group $G/Z(G)$ is non-trivial and abelian then the group G is nilpotent of class 2, and this can be determined again from Theorem 2.7. If $G/Z(G)$ is not abelian then we can iterate the process, construct an ω -tree-automatic presentation of the quotient group $(G/Z(G))/Z(G/Z(G))$ and decide whether this group is (non-trivial and) abelian. If this is the case then G is nilpotent of class 3, and so on.

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